ON PERTURBATION THEORY FOR SPECTRAL OPERATORS

BY

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ABSTRACT

Conditions are given under which a Rellich's perturbation theorem for normal operators on Hilbert spaces may be generalized for spectral operators on Banach spaces.

In a previous paper, [6] Theorem 8, we proved that the strong limit of a sequence of spectral operators commuting with a Boolean algebra of projections of finite multiplicity is a spectral operator provided that their resolution of the identity are uniformly bounded. In connection with this theorem, Foguel has raised the question whether a theorem similar to Rellich's theorem [5] p. 678 (see also [2] Theorem X–7–2) is true in the above mentioned case.

In this note we consider a sequence of spectral operators of scalar type $\{S\}$, converging strongly to a scalar operator S, and we give a necessary and sufficient condition for the sequence $\{f(S_n)\}$ to converge strongly to $f(S)$ for every bounded Borel function f for which the resolution of the identity of S vanishes on a closed set containing the discontinuities of f. The required perturbation theorem for spectral operators commuting with a Boolean algebra of projections of finite multiplicity will be an immediate consequence of the precedent theorem.

Foguel has introduced in [3] p. 59 the notions of real and imaginary parts of a scalar operator. In fact, he has proved that every scalar operator S whose resolution of the identity is $E(\,\cdot\,)$ has a unique decomposition

$$S = R + iJ$$

where $R = \mathfrak{Re}\, S = \int (\mathfrak{Re}\, \lambda)\, E(d\lambda)$ is called the real part of S and $J = \mathfrak{Im}\, S = \int (\mathfrak{Im}\, \lambda)\, E(d\lambda)$ the imaginary part (R and J have real spectrum). Using these concepts and some ideas from the proof of Rellich's theorem for normal operators we get the next basic result.

THEOREM. *Let $\{S_n\}$ be a sequence of scalar operators on a Banach space X converging strongly to a scalar operator S and such that their resolutions of the identity are uniformly bounded i.e.* $\| E(S_n, \cdot) \| \leqq M$; $n = 1, 2, \ldots$ *Then the sequence*

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$\{f(S_n)x_0\}$ converges to $f(S)x$ for every bounded Borel function f and $x_0 \in X$ satisfying $E(S,\delta)x_0 = 0$ (where δ denotes the closure of the set of all discontinuities of f) if and only if $\{R_n\}, R_n = \Re e\, S_n$, converges strongly to $R = \Re e\, S$.

Proof. The necessity is obvious since $\Re e\, \lambda$ is a continuous function. In order to prove the sufficiency, first, let us observe that from our hypotheses it follows

$$\lim_{n\to\infty} (R_n - iJ_n)x = \lim_{n\to\infty} \int \bar{\lambda}\, E(S_n, d\lambda)x = \int \bar{\lambda}\, E(S, d\lambda)x; \; x \in X$$

and further

$$\lim_{n\to\infty} \int p(\lambda,\bar{\lambda})\, E(S_n, d\lambda)x = \int p(\lambda,\bar{\lambda})\, E(S, d\lambda)x; \; x \in X$$

for every polynomial $p(\lambda,\bar{\lambda})$. Now, let Λ be compact set containing the spectra of S and S_n; $n = 1,2,\cdots$ and g a function continuous on Λ. In view of the Stone-Weierstrass theorem there exists a sequence of polynomials $\{p_k(\lambda,\bar{\lambda})\}$ which converges uniformly on Λ to $g(\lambda)$. Thus,

$$\left\| \int [g(\lambda) - p_k(\lambda,\bar{\lambda})]\, E(S_n, d\lambda) \right\| \leqq 4M \sup_{\lambda\in\Lambda} |g(\lambda) - p_k(\lambda,\bar{\lambda})|; \; k, n = 1,2,\cdots$$

$$\left\| \int [g(\lambda) - p_k(\lambda,\bar{\lambda})]\, E(S, d\lambda) \right\| \leqq 4M_1 \sup_{\lambda\in\Lambda} |g(\lambda) - p_k(\lambda,\bar{\lambda})|; \; k = 1,2,\cdots$$

where M_1 is a bound for the resolution of the identity of S. Further, it follows

$$\begin{aligned} &\left\| \int g(\lambda)\, E(S_n, d\lambda)x - \int g(\lambda)\, E(S, d\lambda)x \right\| \leqq \\ &\leqq \left\| \int [g(\lambda) - p_k(\lambda,\bar{\lambda})]\, E(S_n, d\lambda) \right\| \cdot \|x\| + \left\| \int [g(\lambda) - p_k(\lambda,\bar{\lambda})]\, E(S, d\lambda) \right\| \cdot \|x\| + \\ &\quad + \left\| \int p_k(\lambda,\bar{\lambda})\, E(S_n, d\lambda)x - \int p_k(\lambda,\bar{\lambda})\, E(S, d\lambda)x \right\| \underset{n\to\infty}{\to} 0; \; x \in X \end{aligned}$$

for every function g continuous on Λ.

Now, let f be a bounded Borel function on A, δ the closure of the set of all its discontinuities and $x_0 \in X$ such that $E(S,\delta)x_0 = 0$. By Urysohn's lemma [2] I-5-2 there exists, for each $m \geqq 1$, a continuous function u_m with $0 \leqq u_m(\lambda) \leqq 1$; $u_m(\lambda) = 0$ for $\lambda \in \delta$ and $u_m(\lambda) = 1$ when $\min_{\mu\in\rho} |\lambda - \mu| \geqq 1/m$. One can easily see that

$$\lim_{m\to\infty} \int u_m(\lambda)\, E(S, d\lambda)x_0 = x_0$$

Since $f \cdot u_m$ is a continuous function it follows from the first part of the proof that

$$\lim_{n\to\infty} \int f(\lambda)u_m(\lambda)E(S_n,d\lambda)x = \int f(\lambda)u_m(\lambda)E(S,d\lambda)x; \; x \in X; \; m = 1,2,\cdots$$

and in view of the uniform boundedness of $\| E(S_n,\cdot) \|$; $n = 1,2,\cdots$ we have

$$\lim_{n\to\infty} \int f(\lambda)E(S_n,d\lambda)x_0 = \int f(\lambda)E(S,d\lambda)x_0 \qquad \text{Q.E.D.}$$

The present theorem generalizes a theorem of Kaplansky [4] for self-adjoint operators and a theorem of Bade [1] concerning scalar operators with real spectrum.

COROLLARY 1. *Under the hypotheses of the above theorem, if for some Borel set σ and $x_0 \in X$ $E(S$, boundary $\sigma)x_0 = 0$, then*

$$\lim_{n\to\infty} E(S_n,\sigma)x_0 = E(S,\sigma)x_0$$

From these results and [6] Theorem 8 is derived

COROLLARY 2. *Let $\{A_n\}$ be a sequence of spectral operators on a Banach space X commuting with a Boolean algebra of projections of uniform finite multiplicity which converges strongly to an operator A. If there is a constant K such that $\| E(A_n,\delta)\| \leqq K$; $n = 1,2,\cdots,\delta \in$ Borel sets, then A is spectral and*

$$\lim_{n\to\infty} E(A_n,\sigma)x_0 = E(A,\sigma)x_0$$

for every Borel set σ and $x_0 \in X$ satisfying $E(A$, boundary $\sigma)x_0 = 0$.

If the underlying space X is a separable Hilbert space, then using [7] Theorem 13 instead of [6] Theorem 8 we can generalize Corollary 2 for spectral operators commuting with a Boolean algebra of projections containing no projections of infinite uniform multiplicity.

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